Solution by the Proposer.

Inspection of the two equations shows that $m = \frac{1}{2}$, $n = \frac{2}{3}$, and since the first equation is homogeneous in (a, x, y), $k = a^{1/3}$. The given equation is in fact the result of rationalizing

$$y = a^{1/3}x^{2/3} \pm \sqrt{a^2 - x^2}.$$

Let $x/a = \sin\theta$. Then $y/a = \sin^{2/3}\theta \pm \cos\theta$. Hence, $y'/a = 2/3 \sin^{-1/3}\theta \mp \tan\theta$, since $d\theta/dx = \sec\theta$. Hence, $y''/a = -2/9 \sin^{-4/3}\theta \mp \sec^3\theta$. When $y' = 0$, $\tan\theta\cdot\sin^{1/3}\theta = 2/3$. Solving, $\theta_1 = 38° \; 4.2'$, $x_1/a = 0.6166$, and $y_1/a = .7245 \pm .7873 = 1.5117$ or -0.0628. The first value gives a maximum ordinate. When $y'' = 0$, $\tan^{.8}\theta \sec\theta = (2/9)^{.6}$. Solving, $\theta_2 = 17° \; 1'$, $x_2/a = 0.292\bar{7}$, and $y_2/a = .4408 \pm .9562 = 1.3970$ or -0.5154. $y_2/a' = 1.004\bar{1} \mp .306\bar{1} = 0.6981$ or 1.3102.

The second value gives an inflection.

The curve is readily constructed by adding ordinates of the semi-cubical parabola and the circle.

(a, a) is evidently the point of maximum abscissa.

Also solved by Adele Holtwick.

439 (Calculus). Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution by C. C. Yen, Tangshan, North China.

Let $P \equiv (x, y, z)$ be the vertex of the rectangular parallelopiped lying in the first octant of the ellipsoid. Then the volume of the parallelopiped is $V = 8xyz$.

Since P lies on the ellipsoid, the coördinates (x, y, z) satisfy the equation of the ellipsoid, and therefore

$$V = 8xyz = 8cxy(1 - x^2/a^2 - y^2/b^2)^{1/2} = 8c\cdot F(x, y), \tag{1}$$

where $F(x, y) \equiv x\cdot y(1 - x^2/a^2 - y^2/b^2)^{1/2}$ is maximum when and only when V is maximum.

Differentiating, we get

$$\begin{aligned}\frac{\partial F}{\partial x} &= y(1 - 2x^2/a^2 - y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2},\\ \frac{\partial F}{\partial y} &= x(1 - x^2/a^2 - 2y^2/b^2) \div (1 - x^2/a^2 - y^2/b^2)^{1/2}.\end{aligned} \tag{2}$$

Equating to zero the left-hand members of (2), we have

$$2b^2x^2 + a^2y^2 = a^2b^2, \qquad b^2x^2 + 2a^2y^2 = a^2b^2,$$

which give

$$x^2 = \frac{a^2}{3}, \qquad y^2 = \frac{b^2}{3}; \qquad \text{and, therefore,} \qquad x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}.$$

Differentiating (2), we get

$$\frac{\partial^2 F}{\partial x^2} = -\frac{xy}{a^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{2x^2}{a^2} - \frac{3y^2}{b^2}\right), \quad = -\frac{4b}{a\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

$$\frac{\partial^2 F}{\partial y^2} = -\frac{xy}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-3/2}\left(3 - \frac{3x^2}{a^2} - \frac{2y^2}{b^2}\right), \quad = -\frac{4a}{b\sqrt{3}} \text{ when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}};$$

and

$$\frac{\partial^2 F}{\partial y \partial x} = \left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right)\left\{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} + \frac{y^2}{b^2}\right\} - \frac{2y^2}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2},$$

$$= -\frac{2}{\sqrt{3}} \quad \text{when } x = \frac{a}{\sqrt{3}}, \quad y = \frac{b}{\sqrt{3}}.$$

Therefore, when

$$x = \frac{a}{\sqrt{3}}, \qquad y = \frac{b}{\sqrt{3}}, \qquad \Delta \equiv \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial y \partial x}\right)^2 = 4;$$

and since

$$\frac{\partial^2 F}{\partial x^2} < 0, \qquad \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0,$$

we have all the conditions for a maximum value of $F(x, y)$ fulfilled at $(a/\sqrt{3}, b/\sqrt{3})$.

Hence, finally, substituting in (1), we have the volume of the greatest rectangular parallelopiped inscribed in the ellipsoid equal to $8abc/(3\sqrt{3})$.

Also solved by L. E. LUNN, L. E. MENSENKAMP, O. S. ADAMS, C. E. GITHENS, J. L. RILEY, PAUL CAPRON, and H. C. FEEMSTER.

441 (Calculus). Proposed by J. L. RILEY, Stephenville, Texas.

Find the minimum value of

$$\int \left\{ \left(\frac{dy}{dx}\right)^2 \sin x + (y + x - \sin x)^2/\sin x \right\} dx.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

This problem is indefinite and no solution is possible, until the conditions that the end points must satisfy are stated. In fact, if we take $y = mx$ and integrate from $\pi + \epsilon$ to $2\pi - \epsilon$, we can make the integral as small as desired by decreasing ϵ.

It is not difficult to find the equation of the extremals. Euler's differential equation is $d(F_{y'})/dx - F_y = 0$. This becomes for our problem $y'' \sin^2 x + y' \sin x \cos x - y = x - \sin x$. A solution of the equation where we make the right-hand member zero is

$$y = C_1 \tan (x/2) + C_2 \operatorname{ctn} (x/2),$$

obtained by taking $2y'$ as an obvious integrating factor. Completing the solution by any one of several methods, there results

$$y = C_1 \tan (x/2) + C_2 \operatorname{ctn} (x/2) - x + \operatorname{ctn} (x/2) \log \sec^2 (x/2).$$

For any further investigation of the problem, however, a knowledge of the boundary conditions is necessary.

Also solved by ALEXANDER DILLINGHAM.

349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound, over which passes a second string to one end of which is attached a 3-pound weight, and the other end of which is attached to and supports another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights, 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted on by gravity alone show that the accelerations of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{3}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

SOLUTION BY THE PROPOSER.

Let x = distance of 9-pound weight from center of fixed pulley, y = distance of center of P_2 from center of P_1, and z = distance of $3\frac{1}{3}$-pound weight from center of P_2.

Then will $\dot{x}$ = velocity of 9-pound weight, $-\dot{x}$ = velocity of P_1, $\dot{y} - \dot{x}$ = velocity of P_2 $-\dot{y} - \dot{x}$ = velocity of 3-pound weight, $\dot{z} + \dot{y} - \dot{x}$ = velocity of $3\frac{1}{3}$-pound weight, and $-\dot{z} + \dot{y} - \dot{x}$ = velocity of 2-pound weight.